

Non-linear Fractal Interpolating Functions

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We consider two non-linear generalizations of fractal interpolating functions generated from iterated function systems. The first corresponds to fitting data using a K^{th} -order polynomial, while the second relates to the freedom of adding certain arbitrary functions. An escape-time algorithm that can be used for such systems to generate fractal images like those associated with Julia or Mandelbrot sets is also described.

I. INTRODUCTION

An Iterated Function System (IFS) can be used to construct a fractal interpolating function for a given set of data [1, 2]. The simplest such system defines an IFS

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ c_n & 0 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} + \begin{pmatrix} e_n \\ f_n \end{pmatrix}, \quad (1)$$

with coefficients a_n, c_n, e_n , and f_n determined from discrete data points (t_i, x_i) , $i = 0, 1, \dots, N$. Such an IFS interpolates the data set in the sense that, under certain assumptions on the coefficients [1], the attractor of the IFS is a graph that passes through the data points. In this particular case, the IFS can be written as

$$\begin{aligned} t' &= \frac{(t - t_0)}{(t_N - t_0)} t_n + \frac{(t - t_N)}{(t_0 - t_N)} t_{n-1} \\ x' &= \frac{(t' - t_{n-1})}{(t_n - t_{n-1})} x_n + \frac{(t' - t_n)}{(t_{n-1} - t_n)} x_{n-1} \end{aligned} \quad (2)$$

which shows that a linear (in t) interpolating function between the points (t_{n-1}, x_{n-1}) and (t_n, x_n) is used.

Various generalizations of fractal interpolating functions have been given, including those for higher dimensional functions, the use of hidden variables, and extensions to certain non-linear distortions [3, 4, 5, 6, 7]. In this note we describe a generalization whereby the transformation incorporates a K^{th} -order polynomial interpolation between adjacent points. We also discuss certain classes of non-linear functions that can arise in such interpolating functions, and show how such functions can, with the use of a particular escape-time algorithm, be used to generate certain fractal images.

The paper is organized as follows. In Section II we describe simple linear fractal interpolating functions, and discuss how particular non-linear functions can arise. Section III generalizes these considerations to K^{th} -order interpolating functions. Section IV describes a certain escape-time algorithm which may be used for these systems to generate fractal images like those associated with Mandelbrot or Julia sets. Section V contains some brief conclusions.

II. LINEAR INTERPOLATING FUNCTIONS

We first describe how a standard linear fractal interpolating function is constructed. Suppose we have data points (t_i, x_i) , $i = 0 \dots N$, describing a function $x(t)$. Consider the IFS

$$W_n \begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ c_n & 0 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} + \begin{pmatrix} e_n \\ f_n \end{pmatrix} \quad (3)$$

Imposing the conditions, for $n = 1, 2, \dots, N$,

$$W_n \begin{pmatrix} t_0 \\ x_0 \end{pmatrix} = \begin{pmatrix} t_{n-1} \\ x_{n-1} \end{pmatrix}$$

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$$W_n \begin{pmatrix} t_N \\ x_N \end{pmatrix} = \begin{pmatrix} t_n \\ x_n \end{pmatrix} \quad (4)$$

leads to determination of the coefficients as

$$\begin{aligned} a_n &= \frac{t_n - t_{n-1}}{t_N - t_0} \\ e_n &= \frac{t_{n-1}t_N - t_nt_0}{t_N - t_0} \\ c_n &= \frac{x_n - x_{n-1}}{t_N - t_0} \\ f_n &= \frac{x_{n-1}t_N - x_nt_0}{t_N - t_0} \end{aligned} \quad (5)$$

The transformation can then be written as

$$\begin{aligned} W_n(t) \equiv t' &= \frac{(t - t_0)}{(t_N - t_0)} t_n + \frac{(t - t_N)}{(t_0 - t_N)} t_{n-1} \\ W_n(x) \equiv x' &= \frac{(t' - t_{n-1})}{(t_n - t_{n-1})} x_n + \frac{(t' - t_n)}{(t_{n-1} - t_n)} x_{n-1} \end{aligned} \quad (6)$$

Thus, $W_n(x) \equiv x'$ is determined by a linear (in t) interpolating function constructed between the points (t_{n-1}, x_{n-1}) and (t_n, x_n) .

A generalization of this type of fractal interpolating function can be found by considering an IFS of the form

$$\begin{aligned} W_n(t) &= a_nt + e_n \\ W_n(x) &= c_nt + f_n + g_n(x) \end{aligned} \quad (7)$$

where $g_n(x)$ is, at this stage, an arbitrary function. Imposing the conditions (4) leads to determination of the coefficients as

$$\begin{aligned} a_n &= \frac{t_n - t_{n-1}}{t_N - t_0} \\ e_n &= \frac{t_{n-1}t_N - t_nt_0}{t_N - t_0} \\ c_n &= \frac{x_n - x_{n-1}}{t_N - t_0} - \frac{g_n(x_N) - g_n(x_0)}{t_N - t_0} \\ f_n &= \frac{x_{n-1}t_N - x_nt_0}{t_N - t_0} - \frac{g_n(x_0)t_N - g_n(x_N)t_0}{t_N - t_0} \end{aligned} \quad (8)$$

The transformation can then be written as

$$\begin{aligned} W_n(t) \equiv t' &= \frac{(t - t_0)}{(t_N - t_0)} t_n + \frac{(t - t_N)}{(t_0 - t_N)} t_{n-1} \\ W_n(x) \equiv x' &= \frac{(t' - t_{n-1})}{(t_n - t_{n-1})} \tilde{x}_n + \frac{(t' - t_n)}{(t_{n-1} - t_n)} \tilde{x}_{n-1} \end{aligned} \quad (9)$$

where

$$\begin{aligned} \tilde{x}_n &= x_n + g_n(x) - g_n(x_N) \\ \tilde{x}_{n-1} &= x_{n-1} + g_n(x) - g_n(x_0) \end{aligned} \quad (10)$$

and the identity

$$\frac{(t' - t_{n-1})}{(t_n - t_{n-1})} + \frac{(t' - t_n)}{(t_{n-1} - t_n)} = 1 \quad (11)$$

for arbitrary t' has been used.

III. QUADRATIC AND HIGHER ORDER INTERPOLATING FUNCTIONS

The interpolating function of the last section used a linear (in t) approximation between adjacent points. In this section we indicate how a quadratic approximation may be constructed; the generalization to an arbitrary K^{th} -order polynomial approximation will be straightforward. Let us consider a transformation of the form

$$\begin{aligned} W_n(t) &= a_n t + e_n \\ W_n(x) &= c_n t + d_n t^2 + f_n \end{aligned} \quad (12)$$

and impose the conditions, for $n = 2, 3, \dots, N$,

$$\begin{aligned} W_n \begin{pmatrix} t_0 \\ x_0 \end{pmatrix} &= \begin{pmatrix} t_{n-2} \\ x_{n-2} \end{pmatrix} \\ W_n \begin{pmatrix} t_m \\ x_m \end{pmatrix} &= \begin{pmatrix} t_{n-1} \\ x_{n-1} \end{pmatrix} \\ W_n \begin{pmatrix} t_N \\ x_N \end{pmatrix} &= \begin{pmatrix} t_n \\ x_n \end{pmatrix} \end{aligned} \quad (13)$$

The point t_m is determined as

$$t_m = \frac{(t_{n-1} - t_{n-2})}{(t_n - t_{n-2})} t_N + \frac{(t_{n-1} - t_n)}{(t_{n-2} - t_n)} t_0 \quad (14)$$

with corresponding point x_m . The coefficients of the IFS are determined as

$$\begin{aligned} a_n &= \frac{t_n - t_{n-2}}{t_N - t_0} \\ e_n &= \frac{t_N t_{n-2} - t_0 t_n}{t_N - t_0} \\ c_n &= \frac{x_n(t_0^2 - t_m^2) + x_{n-1}(t_N^2 - t_0^2) + x_{n-2}(t_m^2 - t_N^2)}{(t_N - t_0)(t_N - t_m)(t_m - t_0)} \\ d_n &= \frac{x_n(t_m - t_0) + x_{n-1}(t_0 - t_N) + x_{n-2}(t_N - t_m)}{(t_N - t_0)(t_N - t_m)(t_m - t_0)} \\ f_n &= \frac{x_n t_m t_0(t_m - t_0) + x_{n-1} t_N t_0(t_0 - t_N) + x_{n-2} t_N t_m(t_N - t_m)}{(t_N - t_0)(t_N - t_m)(t_m - t_0)} \end{aligned} \quad (15)$$

With this, the transformation can be written as

$$\begin{aligned} W_n(t) \equiv t' &= \frac{(t - t_0)}{(t_N - t_0)} t_n + \frac{(t - t_N)}{(t_0 - t_N)} t_{n-2} \\ W_n(x) \equiv x' &= \frac{(t' - t_{n-1})(t' - t_{n-2})}{(t_n - t_{n-1})(t_n - t_{n-2})} x_n + \frac{(t' - t_n)(t' - t_{n-2})}{(t_{n-1} - t_n)(t_{n-1} - t_{n-2})} x_{n-1} \\ &\quad + \frac{(t' - t_{n-1})(t' - t_n)}{(t_{n-2} - t_{n-1})(t_{n-2} - t_n)} x_{n-2} \end{aligned} \quad (16)$$

which thus uses a quadratic (in t') interpolating function between the points (t_n, x_n) , (t_{n-1}, x_{n-1}) , and (t_{n-2}, x_{n-2}) .

As in the previous section, including an arbitrary function $g_n(x)$ in the IFS transformation via

$$\begin{aligned} W_n(t) &= a_n t + e_n \\ W_n(x) &= c_n t + d_n t^2 + f_n + g_n(x) \end{aligned} \quad (17)$$

is straightforward. The conditions (13) leads to determination of the point t_m of Eq. (14) as before, together with the accompanying point x_m . The transformation itself can be written as

$$\begin{aligned} W_n(t) \equiv t' &= \frac{(t - t_0)}{(t_N - t_0)} t_n + \frac{(t - t_N)}{(t_0 - t_N)} t_{n-2} \\ W_n(x) \equiv x' &= \frac{(t' - t_{n-1})(t' - t_{n-2})}{(t_n - t_{n-1})(t_n - t_{n-2})} \tilde{x}_n + \frac{(t' - t_n)(t' - t_{n-2})}{(t_{n-1} - t_n)(t_{n-1} - t_{n-2})} \tilde{x}_{n-1} \\ &\quad + \frac{(t' - t_{n-1})(t' - t_n)}{(t_{n-2} - t_{n-1})(t_{n-2} - t_n)} \tilde{x}_{n-2} \end{aligned} \quad (18)$$

where

$$\begin{aligned}\tilde{x}_n &= x_n + g_n(x) - g_n(x_N) \\ \tilde{x}_{n-1} &= x_{n-1} + g_n(x) - g_n(x_m) \\ \tilde{x}_{n-2} &= x_{n-2} + g_n(x) - g_n(x_0)\end{aligned}\tag{19}$$

and the identity

$$\frac{(t' - t_{n-1})(t' - t_{n-2})}{(t_n - t_{n-1})(t_n - t_{n-2})} + \frac{(t' - t_n)(t' - t_{n-2})}{(t_{n-1} - t_n)(t_{n-1} - t_{n-2})} + \frac{(t' - t_{n-1})(t' - t_n)}{(t_{n-2} - t_{n-1})(t_{n-2} - t_n)} = 1\tag{20}$$

for arbitrary t' has been used.

From these considerations, the pattern to constructing a K^{th} -order fractal interpolating function is apparent. Start with a transformation of the form

$$\begin{aligned}W_n(t) &= a_n t + e_n \\ W_n(x) &= B_n^{(0)} + B_n^{(1)}t + B_n^{(2)}t^2 + \dots + B_n^{(K)}t^K\end{aligned}\tag{21}$$

and impose the conditions, for $n = K, K+1, \dots, N$,

$$\begin{aligned}W_n \begin{pmatrix} t_0 \\ x_0 \end{pmatrix} &= \begin{pmatrix} t_{n-K} \\ x_{n-K} \end{pmatrix} \\ W_n \begin{pmatrix} t_{m1} \\ x_{m1} \end{pmatrix} &= \begin{pmatrix} t_{n-K+1} \\ x_{n-K+1} \end{pmatrix} \\ W_n \begin{pmatrix} t_{m2} \\ x_{m2} \end{pmatrix} &= \begin{pmatrix} t_{n-K+2} \\ x_{n-K+2} \end{pmatrix} \\ &\vdots \\ W_n \begin{pmatrix} t_N \\ x_N \end{pmatrix} &= \begin{pmatrix} t_n \\ x_n \end{pmatrix}\end{aligned}\tag{22}$$

The $K-1$ intermediate points t_{mj} , with $j = 1, 2, \dots, K-1$, are determined as

$$t_{mj} = \frac{(t_{n-K+j} - t_{n-K})}{(t_n - t_{n-K})} t_N + \frac{(t_{n-K+j} - t_n)}{(t_{n-K} - t_n)} t_0\tag{23}$$

along with the corresponding x_{mj} points. The resulting transformation will be of the form given by Lagrange's formula for a K^{th} -order polynomial interpolating function constructed from $K+1$ points:

$$\begin{aligned}W_n(t) \equiv t' &= \frac{(t - t_0)}{(t_N - t_0)} t_n + \frac{(t - t_N)}{(t_0 - t_N)} t_{n-K} \\ W_n(x) \equiv x' &= \frac{(t' - t_{n-1})(t' - t_{n-2}) \cdots (t' - t_{n-K})}{(t_n - t_{n-1})(t_n - t_{n-2}) \cdots (t_n - t_{n-K})} x_n + \\ &+ \frac{(t' - t_n)(t' - t_{n-2}) \cdots (t' - t_{n-K})}{(t_{n-1} - t_n)(t_{n-1} - t_{n-2}) \cdots (t_{n-1} - t_{n-K})} x_{n-1} + \dots + \\ &+ \frac{(t' - t_n)(t' - t_{n-1}) \cdots (t' - t_{n-K+1})}{(t_{n-K} - t_n)(t_{n-K} - t_{n-1}) \cdots (t_{n-K} - t_{n-K+1})} x_{n-K}\end{aligned}\tag{24}$$

The inclusion of an arbitrary function $g_n(x)$ in the transformation $W_n(x)$ of Eq. (22), as was done for the linear and quadratic transformations of Eqs. (7) and (17) respectively, is straightforward. As might be expected, the use of these higher-order interpolating functions can increase the accuracy of the interpolation significantly, at least for smooth functions – some informal tests on known functions suggest an improvement of almost an order of magnitude in general in using a quadratic interpolating function over a linear one. Of course, as for polynomial interpolation, there is a limit to the net gain in employing a higher-order interpolating function.

IV. ESCAPE-TIME ALGORITHM

Assuming that the corresponding IFS transformation is contractive, so that the distance $d(t, x)$ between any two points in the range of interest satisfies

$$d(W_n(t), W_n(x)) \leq s_n d(t, x), \quad (25)$$

where $0 < s_n \leq 1$ is the contractivity factor, graphs of the functions represented by fractal interpolating functions can be made by applying the standard random iteration algorithm to the IFS:

- initialize (t, x) to a point in the interval of interest
- for a set number of iterations
 - randomly select a transformation $W_n(t, x)$
 - plot $(t', x') = W_n(t, x)$
 - set $(t, x) = (t', x')$
- end for

Alternatively, one can relate an IFS $W_n(t, x)$ to a shift dynamical system $f(t, x)$, and on this system perform an escape time algorithm to generate an image [1]. In this section we describe an algorithm for generating fractal images like those for Julia or Mandelbrot sets from IFS interpolating functions.

Suppose we have an IFS transformation $W_n(t, x)$, generated by some data points (x_i, y_i) , $i = 0, 1, \dots, N$, which includes a non-linear function $g_n(x)$, as was done for the linear and quadratic transformations of Eqs. (7) and (17) respectively. We now continue the real variable x of this transformation to complex values: $x \rightarrow z = (z_R, z_I)$, so that the transformation $W_n(t, z)$ is defined on the complex plane. We can then, in analogy with the algorithm used for Julia sets, define the following escape-time algorithm to generate a fractal pattern:

- for each pixel in a region of interest
 - initialize t
 - initialize $z = (z_R, z_I)$ to the pixel coordinates
 - for $n = 0, 1, \dots, N$
 - * calculate $(t', z') = W_n(t, z)$
 - * break if $\sqrt{z_R'^2 + z_I'^2}$ exceeds a maximum
 - * set $(t, z) = (t', z')$
 - end for
 - plot the pixel
- end for

where the pixel is plotted using a coloring algorithm based upon, amongst perhaps other factors, the number of iterations attained when the break condition was met [8].

The preceding can be interpreted as follows. A general K^{th} -order fractal interpolating IFS

$$\begin{aligned} W_n(t) &= a_n t + e_n \\ W_n(x) &= B_n^{(0)} + B_n^{(1)} t + B_n^{(2)} t^2 + \dots + B_n^{(K)} t^K + g_n(x), \end{aligned} \quad (26)$$

with the coefficients $a_n, e_n, B_n^{(0)}, B_n^{(1)}, \dots, B_n^{(K)}$ determined from the data (t_i, x_i) , $i = 0, 1, \dots, N$, can be viewed, with the continuation $x \rightarrow z$, as defining a complex map

$$\begin{aligned} t_{n+1} &= a_n t_n + e_n \\ z_{n+1} &= B_n^{(0)} + B_n^{(1)} t_n + B_n^{(2)} t_n^2 + \dots + B_n^{(K)} t_n^K + g_n(z_n) \end{aligned} \quad (27)$$

for $n = 0, 1, \dots, N$. The escape-time algorithm described above is then just the standard one used for Julia sets of complex maps.

The arbitrariness of the function $g_n(x)$ and the data set (t_i, x_i) used to fix the IFS leads to a wide variety of possible fractal images generated in this way. An interesting class of functions $g_n(x)$ to consider in this context are those for which, when continued to the complex plane $x \rightarrow z = (z_R, z_I)$, lead to a map having a fixed point $z_I^* = 0$:

$$z_I^* = 0 = \text{Im}g_n(z_R, z_I^*) \quad (28)$$

In such a case one could augment the usual condition of the escape-time algorithm to cease iteration: $\sqrt{z_R^2 + z_I^2} > \Lambda$, where Λ is some suitably large value, to also cease iteration when $|z_I| < \epsilon$, where ϵ is a suitably small value. The coloring algorithm used to plot a pixel, which depends on the number of iterations attained when this break-out condition was met (if at all), will then lead to structure in the region where the break-out condition on the magnitude of z is not met.

We give two examples of fractal images generated this way for the choice $g_n(x) = -0.4x + 0.5x^2 + 0.2x^3$, with the data generated from the logistic map $x_{n+1} = 3.4x_n(1 - x_n)$, with $n = 0, 1, \dots, 60$. The first one, appearing in Fig. 1, corresponds to the generalization (7) of a linear (in t) fractal interpolating function, while the second image of Fig. 2 corresponds to the generalization (17) of a quadratic (in t) interpolating function. A coloring algorithm that simply mapped a color to the number of iterations attained when the break-out condition became satisfied was used in both cases.

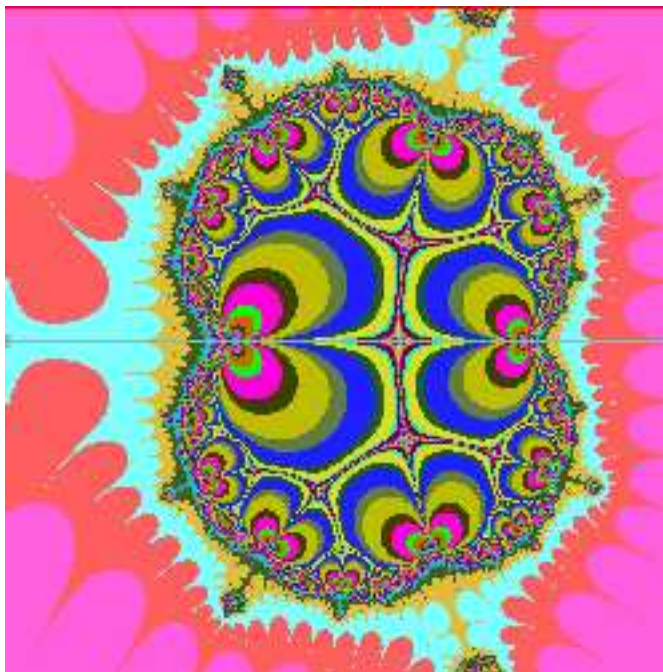


FIG. 1: Fractal image from a t -linear interpolating function

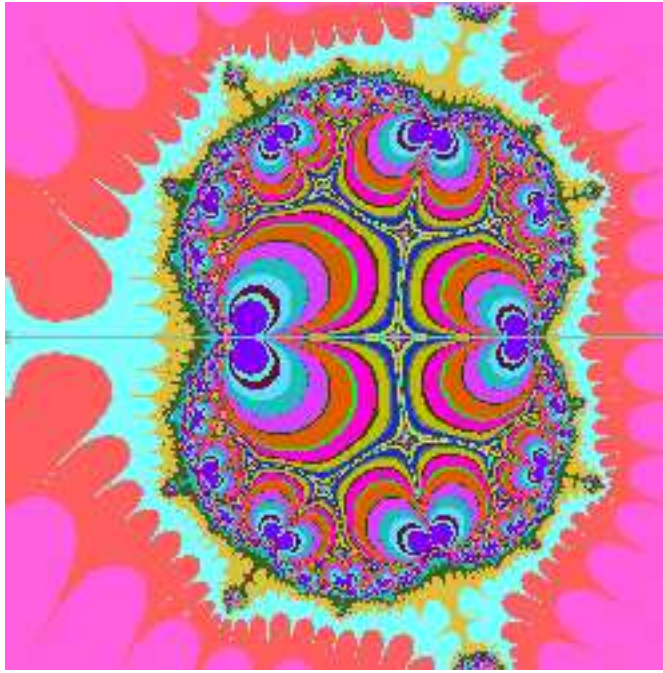


FIG. 2: Fractal image from a t -quadratic interpolating function

These figures illustrate, in the interior of the fractal object, the richer structure arising from the quadratic over the linear interpolation function. In this region the break-out condition $|z_I| < \epsilon$ is satisfied, which numerically for $\epsilon \sim 10^{-5}$ is attained after a relatively small number (10–30) of iterations.

V. CONCLUSIONS

We have considered two non-linear generalizations of fractal interpolating functions constructed from iterated function systems. One – using a K^{th} -order interpolating polynomial – can potentially improve the accuracy of fractal interpolating functions. The other generalization – the use of certain arbitrary functions in the IFS – can, together with an appropriate escape-time algorithm, generate fractal images. This last point is of interest as, first of all, there is a rich variety of such images possible due to the arbitrariness of the functions used, and secondly, it shows how fractal images as normally associated with Julia or Mandelbrot sets can also be associated with discrete data sets.

Acknowledgments

This work was supported by the Natural Sciences and Engineering Research Council of Canada.

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